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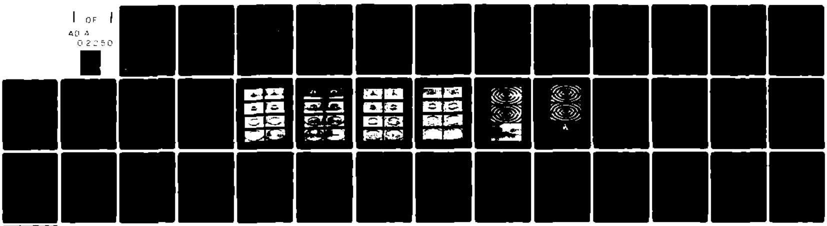

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DR. NORBERT N. BOJARSKI

Research Contractor and Consultant
to the DEPARTMENT OF DEFENSE

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July 1980

Prepared for and Supported by

The Office of Naval Research
800 North Quincy Street
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ABSTRACT

The exact inverse scattering integral equation of this author is applied to passive seismic artillery location in unknown media. It is shown that for such media this integral equation need not be solved, since a time-space display of the effectal field spatially and temporally locates the impulsive sources. Numerico-experimental verification is presented.

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SECTION I

INTRODUCTION

In section II the Rayleigh surface wave equation is reformulated into an inhomogeneous free-space wave equation with a source term depending on an arbitrarily chosen reference velocity. In section III the exact inverse scattering integral equation of this author is rederived and applied to this inhomogeneous free-space Rayleigh surface wave equation. The physical meaning of the effectal field is discussed in section IV. Specifically, that this effectal field consists of the destructive interference between the true causal field and a fictitious anti-causal field, both propagating at the reference velocity, and that if the true field contains impulsive sources, then the effectal field also contains impulsive sources at the same spatial and temporal locations. A graphic time-space display of the effectal field thus yields the correct temporal and spatial locations of the impulsive sources, obviating the need for a solution of the inverse scattering integral equation for the case where only these locations are needed, and the unknown medium velocity is not needed. In section V the two dimensional inverse scattering integral equation is reformulated in the time domain; further illustrating the physical meaning of the effectal field. Numerico-experimental results are presented in section VI.

A closed form solution of a more general exact inverse scattering integral equation, developed under a separate contract, is presented in the appendix section IX. Numerico-experimental verification of this solution is also presented in section VI.

In light of these numerico-experimental results, recommendations for future research are made in section VII.

SECTION II

REFORMULATION OF THE RAYLEIGH SURFACE WAVE EQUATION

The two-dimensional Rayleigh surface wave equation in the frequency domain is

$$\nabla^2\phi + k_r^2 \phi = 0 \quad , \tag{1}$$

where

$$k_r = \frac{\omega}{v_r(\mathbf{x},\omega)} \quad , \tag{2}$$

and where the velocity v_r is related to the shear modulus μ and the mass density σ by

$$v_r = 0.9194 \sqrt{\frac{\mu}{\sigma}} \quad . \tag{3}$$

Next, let a potential V and a source density ρ be defined respectively in terms of an arbitrarily chosen reference velocity c as

$$V(\mathbf{x},\omega) = \frac{\omega^2}{c^2} - \frac{\omega^2}{v_r^2} \quad , \tag{4}$$

$$\rho(\mathbf{X},\omega) = V(\mathbf{X},\omega)\ \phi(\mathbf{X},\omega) \quad . \tag{5}$$

The wave equation (1) thus becomes the inhomogeneous wave equation

$$\nabla^2\phi + k^2\phi = -\rho \quad , \tag{6}$$

subject to the constitutive equation (4).

SECTION III

THE INVERSE SCATTERING INTEGRAL EQUATION

In this section the inverse scattering integral equation of this author [1] will be rederived. This integral equation has been discussed and studied extensively by this author [2] - [7] and others [8] - [28].

Let a field ϕ satisfy the inhomogeneous wave equation

$$\nabla^2 \phi + k^2 \phi = -\rho \quad . \tag{7}$$

Furthermore, let G be the free-space (relative to v=c) Green's function for (7), which satisfies the inhomogeneous wave equation

$$\nabla^2 G + k^2 G = -\delta \quad , \tag{8}$$

and the Sommerfeld radiation condition at infinity.

next, let H be the imaginary part of the Green's function G; i.e.,

$$H \equiv Im\, G \quad . \tag{9}$$

It thus follows from (8) that this imaginary part of the Green's function satisfies the homogeneous wave equation

$$\nabla^2 H + k^2 H = 0 \quad . \tag{10}$$

Next, let the effectal field θ (the term "effectal" will be explained and justified in section IV) be defined as

$$\theta \equiv \int_S d\mathbf{S} \cdot (\phi \; \nabla H - H \; \nabla \phi) \quad , \tag{11}$$

where S is the surface on which the field ϕ is known, and where the support of the unknown sources ρ is inside this surface S.

By Green's theorem, (11) reduces to

$$\theta = \int_V dv (\phi \; \nabla^2 H - H \; \nabla^2 \phi) \quad , \tag{12}$$

which, by (8) and (10), further reduces to

$$\theta = \int_V dv \; [\; \phi \; (-k^2 H) - H \; (-k^2 \phi - \rho) \;] \quad , \tag{13}$$

$$\theta = \int_V dv \; H \; \rho \quad . \tag{14}$$

The unknown sources ρ in the volume V are thus related to the known effectal field θ in the volume V by the proper (i.e., x and $x' \epsilon$ V) Fredholm integral equation of the first kind; where the effectal field θ in the volume V is computable from knowledge of the field ϕ on the surface S by (11).

SECTION IV

THE PHYSICAL MEANING OF THE EFFECTAL FIELD

The imaginary part of the Green's function can be written as

$$H = \frac{1}{2i} (G - G^{*}) \quad . \tag{15}$$

Since the Green's function is causal, it follows from the Fourier transform relationship

$$G(r,\omega r/c) \leftrightarrow g(r,t-r/c) \tag{16}$$

that

$$G^{*}(r,\omega r/c) \leftrightarrow g(r,t+r/c) \quad . \tag{17}$$

It thus follows from (15) that

$$H(r,\omega) \leftrightarrow \frac{1}{2i} [g(r,t-r/c) - g(r,t+r/c)] \quad . \tag{18}$$

In the time domain, the effectal field Φ is thus by (14) and (18)

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi}\int_V dv\ [\ q(r,t-r/c) - q(r,t+r/c)\ ]\star\rho(\mathbf{x},t)\ . \qquad (19)$$

The first convolution on the right hand side of (19) is clearly the real physical time-retarded causal field $\phi(x,t)$ radiated by the sources $\rho(x,t)$. The second integral on the right hand side of (19) is, however, a fictitious time-advanced anti-causal field "radiated" by the sources $\rho(x,t)$. The effectal field $\Phi(x,t)$ is thus an imaginary field due to the destructive interference between the time-retarded causal field and the fictitious time-advanced anti-causal field.

Examination of (4), (5), and (19) readily reveals that if the field contains spatially and temporally impulsive sources, then the effectal field will also contain these impulsive sources at the correct spatial and temporal locations. A graphic time-space display of the effectal field will thus yield these correct spatial and temporal locations, obviating the need for a solution of the inverse scattering integral equation (19) for the case where only these locations are needed.

SECTION V

THE TWO-DIMENSIONAL EFFECTAL FIELD IN THE TIME DOMAIN

In the frequency domain, the two-dimensional Green's function is

$$G = \frac{i}{4} H_o^{(1)}(kr) \tag{20}$$

$$= \frac{i}{4} \left[J_o(kr) + iY_o(kr) \right] \tag{21}$$

$$= -\frac{1}{4} Y_o(kr) + \frac{i}{4} J_o(kr) \quad . \tag{22}$$

Thus, in the frequency domain, the imaginary part of the two-dimensional Green's function is

$$H = Im\, G \tag{23}$$

$$= \frac{1}{4} J_o(kr) \quad . \tag{24}$$

In the time domain, the "imaginary part" of the two-dimensional Green's function becomes

$$h = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \frac{1}{4} J_o\left(\frac{\omega r}{c}\right) d\omega \quad , \tag{25}$$

which, with the aid of the integral representation of the Bessel functions

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{iz \cos \phi} \cos n\phi \, d\phi \tag{26}$$

yields

$$h = \frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_0^{\pi} e^{-i\omega(t - \frac{r}{c}\cos \phi)} \, d\phi \, d\omega \quad , \tag{27}$$

which, in turn, with the aid of the delta function representation

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} \, d\omega \tag{28}$$

reduces to

$$h = \frac{1}{4\pi}\int_0^{\pi} \delta(t - \frac{r}{c}\cos \phi) \, d\phi \quad . \tag{29}$$

Introducing the change of variable

$$z = \frac{r}{c}\cos \phi \tag{30}$$

yields for (29)

$$h = \frac{1}{4\pi}\int_{-\frac{r}{c}}^{\frac{r}{c}} \frac{\delta(t-z)}{\sqrt{(\frac{r}{c})^2 - z^2}} \, dz \quad , \tag{31}$$

which yields

$$h = \begin{cases} \dfrac{1}{4\pi} \dfrac{1}{\sqrt{(\frac{r}{c})^2 - t^2}} & ; \quad t \in (-\frac{r}{c}, \frac{r}{c}) \\ 0 & ; \quad t \notin (-\frac{r}{c}, \frac{r}{c}) \end{cases} \quad . \tag{32}$$

It is now of interest to compare (32) with the full two-dimensional Green's function in the time domain given by [29]

$$\begin{cases} \dfrac{1}{4\pi} \dfrac{1}{\sqrt{t^2 - (\frac{r}{c})^2}} & ; \quad t > \frac{r}{c} \\ 0 & ; \quad t < \frac{r}{c} \end{cases} \quad . \tag{33}$$

In the time domain, the two-dimensional inverse scattering integral equation (14) is thus

$$\int_{v'} \int_{t'} \eta(\mathbf{x}-\mathbf{x}', t-t') \, \rho(\mathbf{x}', t') \, dt' \, dv' = \theta(\mathbf{x}, t) \quad . \tag{34}$$

As previously indicated, examination of (4), (5), and (34) readily reveals that if the field contains spatially and temporally impulsive sources, then the effectal field will also contain these impulsive sources at the correct spatial and temporal locations. A graphic time-space display of the effectal field will thus yield these correct spatial and temporal locations, obviating the need for a solution of the inverse scattering integral equation (34) for the case where only these locations are needed.

SECTION VI

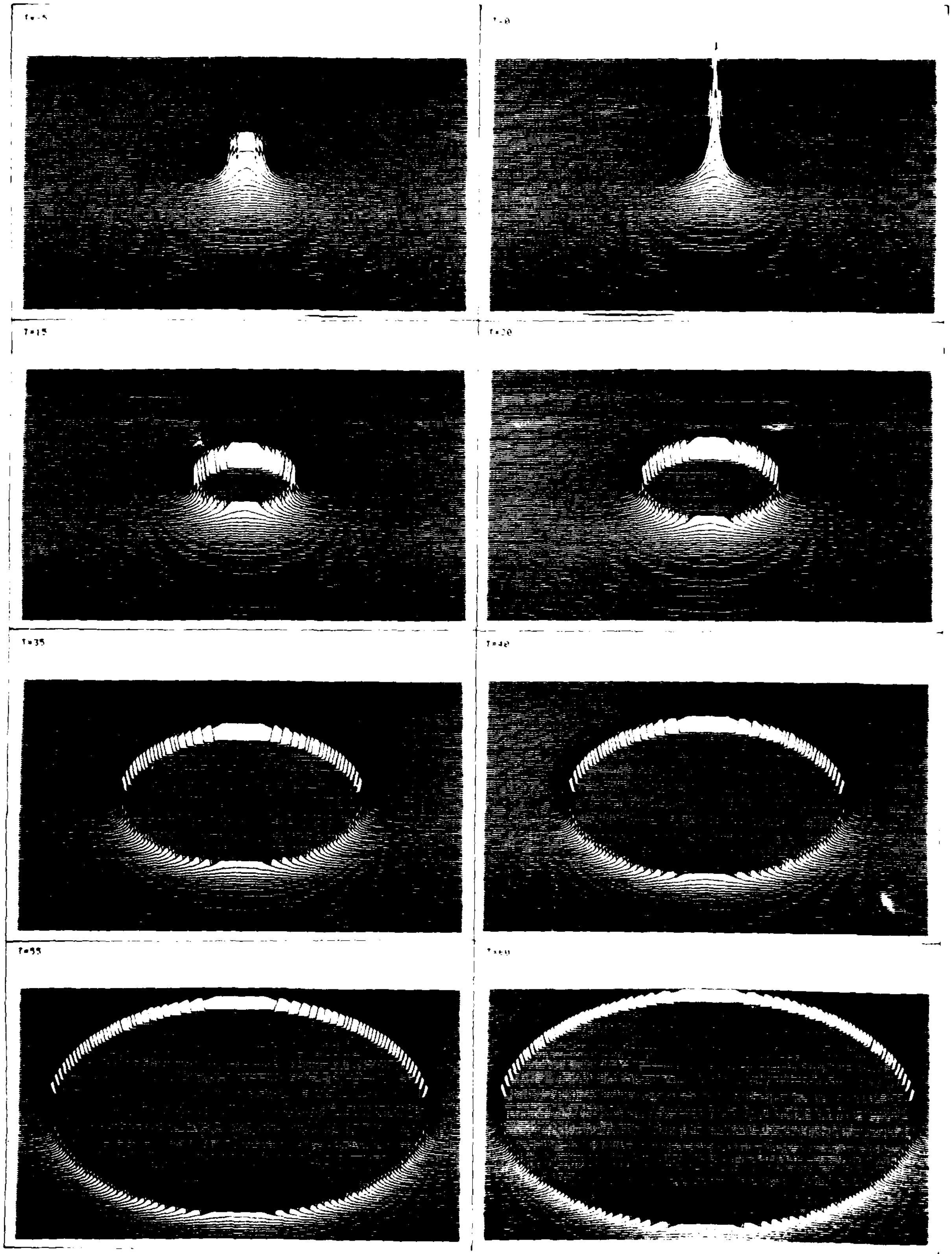
NUMERICO-EXPERIMENTAL RESULTS

A two-dimensional surface with a unity normalized Rayleigh surface wave velocity was selected. A spatially and temporally impulsive source was placed in this surface, and an analytic expression for the radiated field was obtained for all points on the boundary of a 128x128 data grid containing this source. The effectal field at all points in the grid was numerically computed by (11) for 256 frequencies, with a reference velocity c=2 (i.e., 100% off the correct velocity). Next, a fast Fourier transform of the effectal field into the time domain was taken. A graphic time-space representation of every fifth point for t=-5 to t=70 of this effectal field is shown on page 13. As shown in section V, the correct temporal and spatial locations of the impulsive source are obtained (see t=0 frame); however, as anticipated, the effectal field "flies in" and "flies out" at twice the correct velocity.

The preceeding numerical experiment was repeated with a random zero db. signal to noise added to the radiated field prior to the computation of the effectal field by (11). A similar graphic time-space representation for this computation is shown on page 13. Again, the correct temporal and spatial location of the impulsive source is obtained (see t=0 frame), and again, as anticipated, the noise corrupted but recognizable effectal field "flies in" and "flies out" at approximately twice the correct velocity.

When this research was initiated, only the 1973 inverse scattering integral equation of this author was in existence, and only the implementation of the graphic time-space representation of the effectal field was anticipated. A more complete inverse scattering integral equation and its analytic closed form solution (presented in the appendix) were obtained (under separate contract N00014-76-C-0082) in the latter part of this research. This solution was numerico-experimentally verified for a point source and

a point source pair separated by 1/5 of a wavelength, under the same conditions as the time-space representations of pages 13 and 14, except that the implied Hilbert transform was executed with the aid of the Wiener-Lee transform [30]. A graphic representation of the resulting fields is on pages 15 and 16.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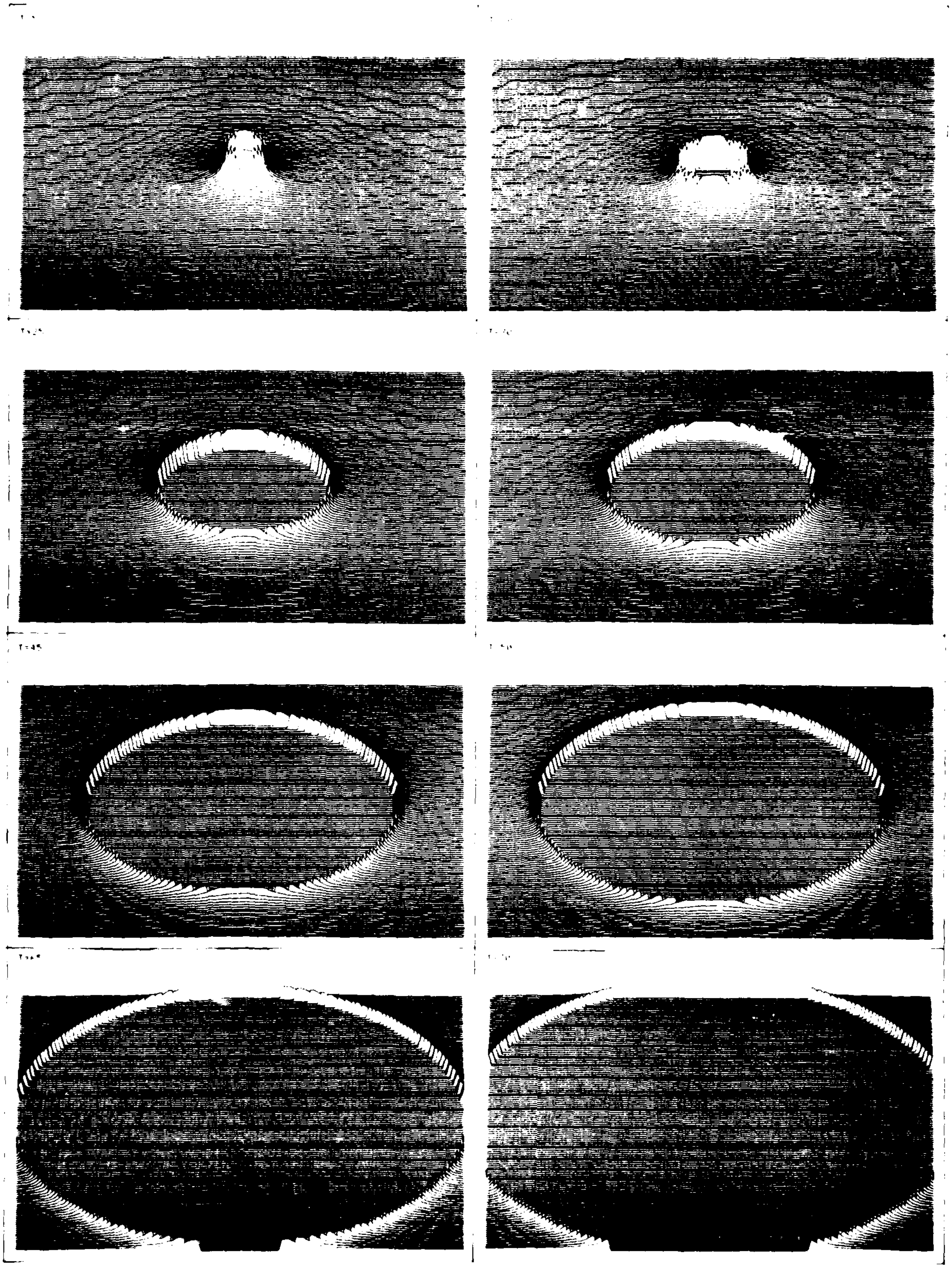

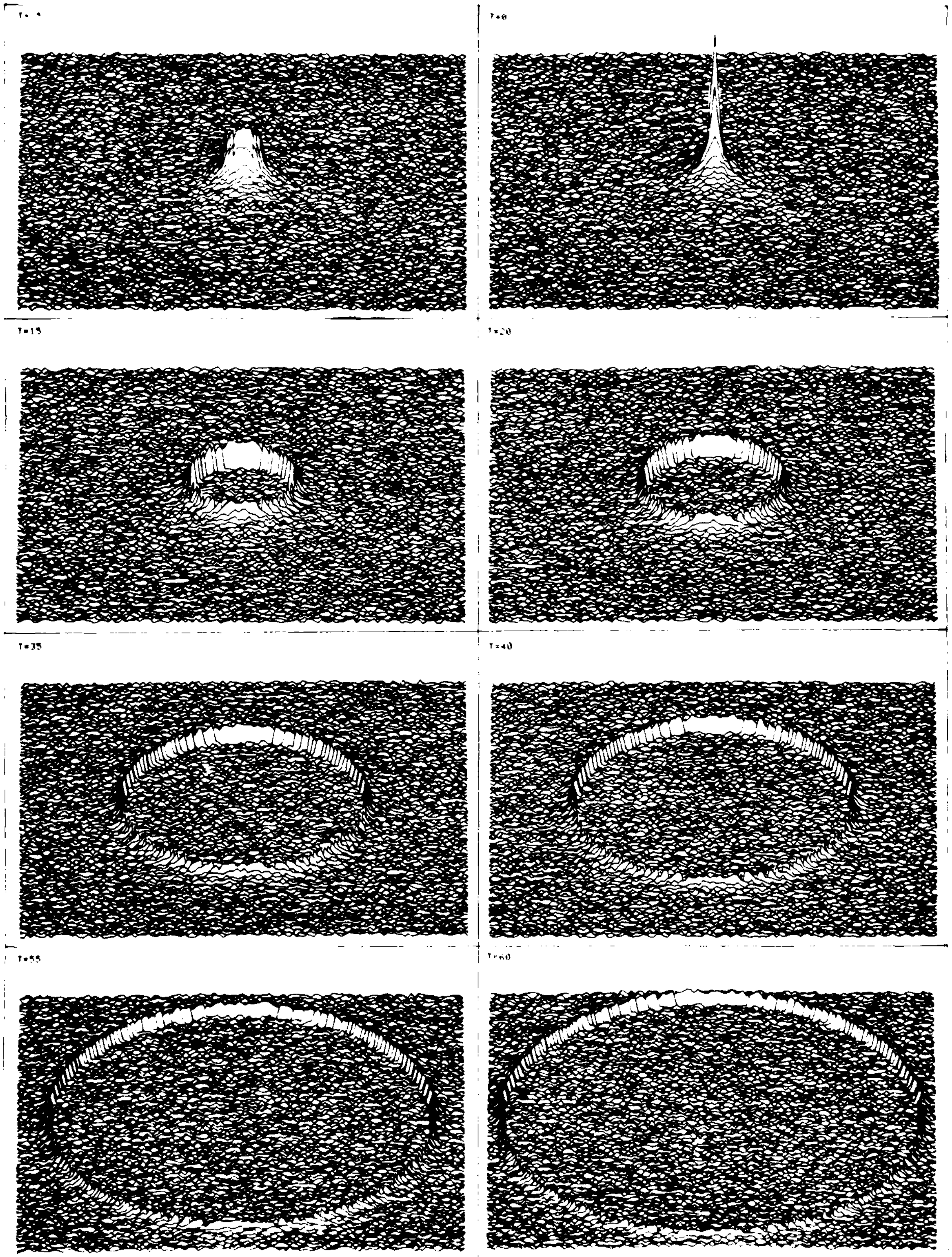
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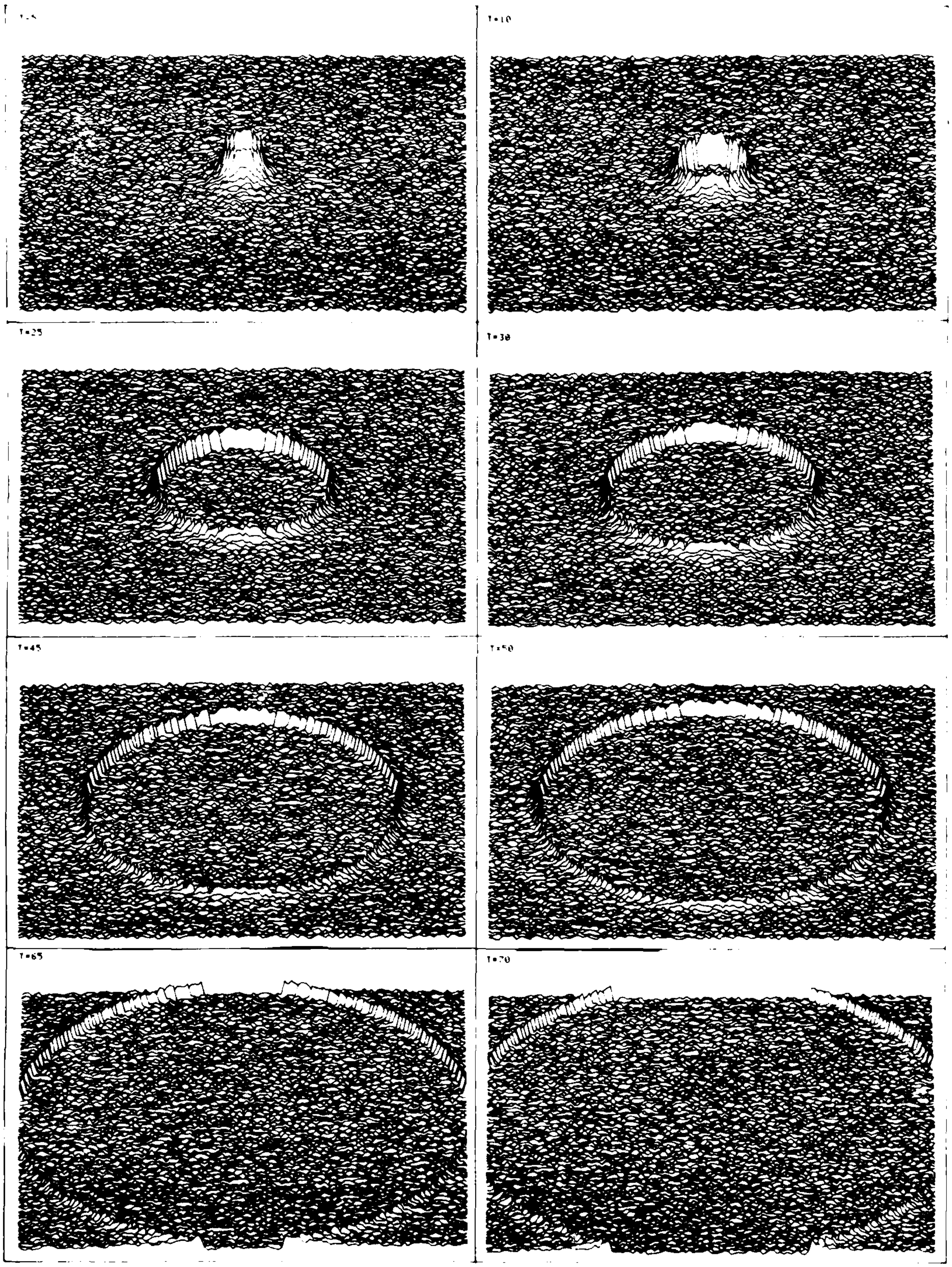
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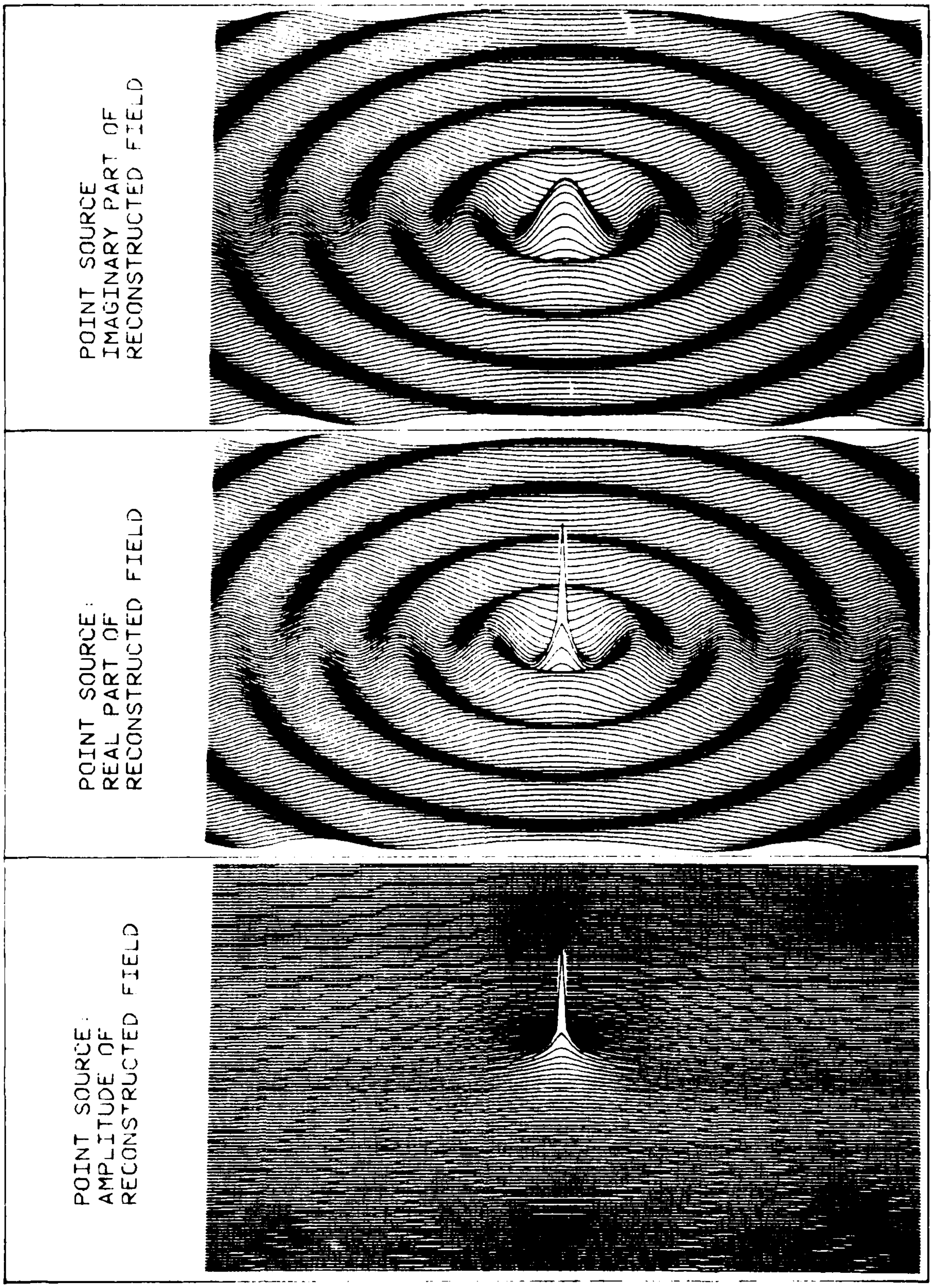
POINT SOURCE:
AMPLITUDE OF
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REAL PART OF
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POINT SOURCE
IMAGINARY PART OF
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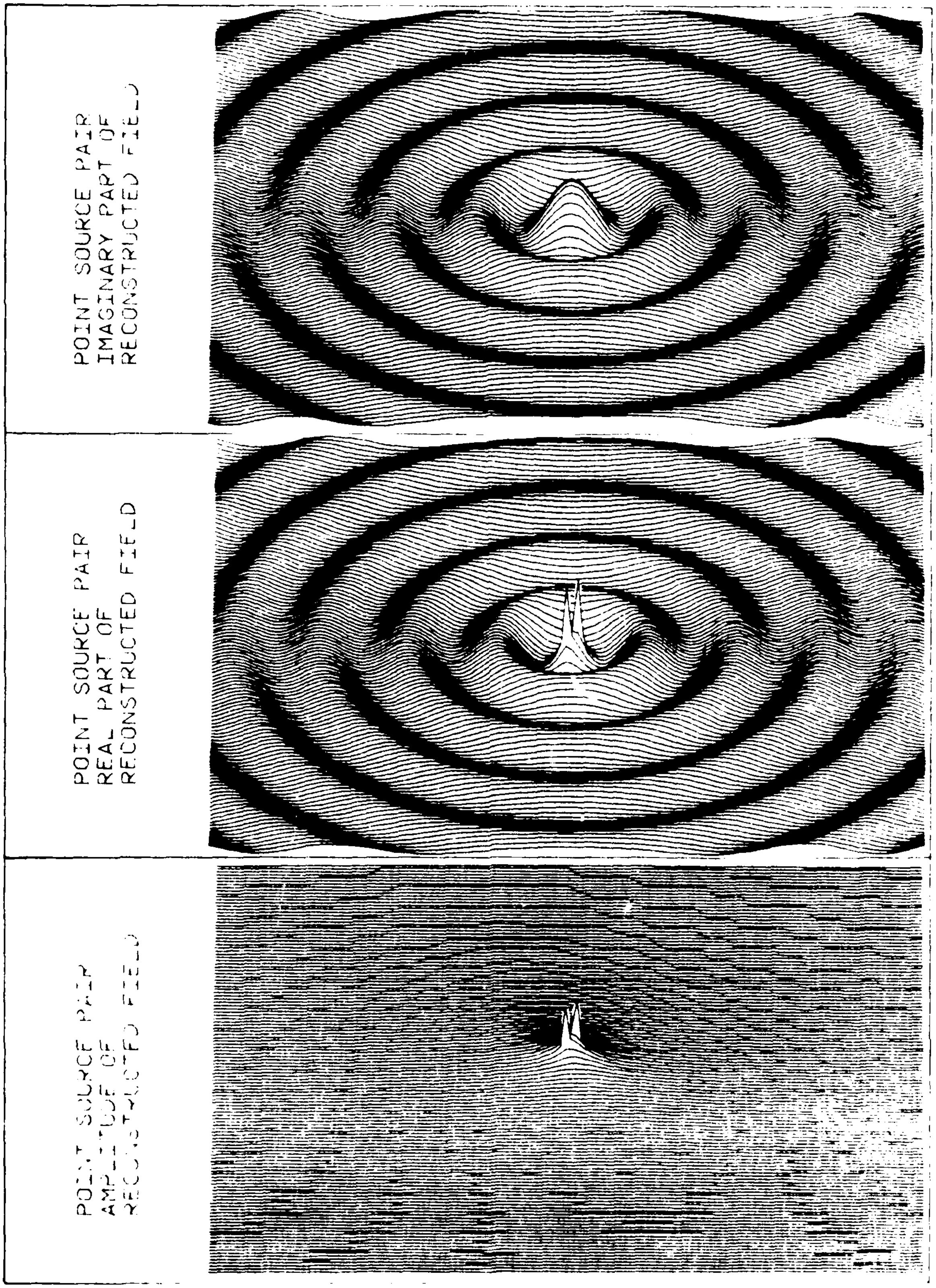
POINT SOURCE PAIR
AMPLITUDE OF
RECONSTRUCTED FIELD
POINT SOURCE PAIR
REAL PART OF
RECONSTRUCTED FIELD
POINT SOURCE PAIR
IMAGINARY PART OF
RECONSTRUCTED FIELD

SECTION VII

RECOMMENDATIONS FOR FUTURE RESEARCH

When this research was initiated, only the 1973 inverse scattering integral equation of this author was in existence, and only the implementation of the graphic time-space representation of the effectal field was anticipated. The more complete inverse scattering integral equation and its analytic closed form solution presented in the appendix were obtained (under separate contract N00014-75-C-0082) only in the latter part of this research. The successful numerico-experimental verification of this new closed form solution indicates that future research should be concentrated toward this solution; specifically, but not nescessarily limited to:

1. Investigate and determine which of the various existing numerical Hilbert transform and Wiener-Lee transform algorithms is best suited for the solution.

2. Investigate and determine the effects of reference slowness sampling density and domain truncation.

3. Investigate and determine the effects of incomplete aspect angle information and and input data sampling density.

4. Investigate and determine the effects of input data noise and errors.

5. Make use of the availability of wide frequency band and/or time domain information, for the purpose of overdetermining the problem and reducing the effects of items 1 through 4.

6. Make use of item 5 when a priori information exists that the medium is non-dispersive.

7. Investigate the feasibility and desirability of casting the exact closed form solution in the time domain.

8. Extend and generalize the exact closed form solution to tensor field for the full elastic wave equation.

SECTION VIII

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SECTION IX

APPENDIX

EXACT INVERSE SCATTERING THEORY

NOTICE: This appendix consists of the reproduction in its entirety of this author's report "Exact Inverse Scattering Theory", March 1980, which was prepared for and supported by the Office of Naval Research, 800 North Quincy Street, Arlington, Virginia 22217, Under contract N00014-76-C-0082.

APPENDIX ABSTRACT

The concepts of reference wave slowness (reciprocal of velocity) and an associated free reference space Green's fuction slowness spectrum are introduced. A modified Kirchhoff surface integral, containing only the imaginary part of this free reference space Green's function slowness spectrum, is formulated, yielding an integral equation for the unknown fields and sources in the interior of a closed surface on which the (remotely sensed) fields are known. A well-posed, analytic closed form solution of this integral equation is obtained.

SECTION A-I

INTRODUCTION

Presented is a unified approach and solution to the inverse scattering and inverse source problems for the inhomogeneous scalar wave equation

$$\nabla^2\phi + \frac{\omega^2}{c^2}\phi = -\rho \quad , \tag{1}$$

subject to the constitutive equation

$$\rho = V\phi \quad , \tag{2}$$

and the homogeneous scalar wave equation

$$\nabla^2\phi + \frac{\omega^2}{c^2(\mathbf{X},\omega)}\phi = 0 \quad . \tag{3}$$

To this end, the single mixed scalar wave equation

$$\nabla^2\phi + \frac{\omega^2}{c^2(\mathbf{X},\omega)}\phi = -\rho \tag{4}$$

is introduced. From an inverse scattering inverse source perspective, (4) reduces to (1) if the medium wave velocity $c(\mathbf{X},\omega)$ is a known constant and the source ρ is the

unknown, and (4) reduces to (3) if the sources [illegible] are known to be zero and the medium wave velocity [illegible]($\mathbf{x}$,[illegible]) is the unknown.

It is argued that the inverse solution presented is an alternative (to the direct Kirchhoff) integration of the wave equation. It is thus appropriate to review [illegible] relevant properties of the direct Kirchhoff integration

$$\int dv \; [\text{illegible}] + \oint d\mathbf{S}\cdot [\text{illegible}] = [\text{illegible}]$$

of the wave equation (1). Specifically, the surface integral in (5) is an equivalence statement relating the field at a field point on one side of the closed surface produced by all the sources on the other side of the closed surface, via the fields produced by these sources on this closed surface. The inverse scattering inverse source problem is, however, characterized by both the field point for the unknown fields as well as all the unknown sources being on the same side of the closed surface (on which the remote sensing is accomplished), for which situation the Kirchhoff surface integral vanishes, thus rendering this Kirchhoff surface integral useless for the inverse scattering inverse source problem. A modified Kirchhoff surface integral, which does not suffer from this pathology, is introduced next.

SECTION A-II

THE INVERSE SCATTERING INTEGRAL EQUATION

Let G be the free reference space Green's function satisfying the inhomogeneous *wave equation*

$$\nabla^2 G + \frac{\omega^2}{v^2} G = -\delta \tag{6}$$

and the Sommerfeld radiation condition at infinity, where v is any arbitrarily chosen reference velocity.

Next, let an effectal field θ be defined as

$$\theta \equiv \oint_S d\mathbf{S} \cdot (\phi \nabla G_i - G_i \nabla \phi) \quad , \tag{7}$$

where

$$G_i \equiv Im\, G \quad , \tag{8}$$

which, by (6), satisfies the homogeneous wave equation

$$\nabla^2 G_i + \frac{\omega^2}{v^2} G_i = 0 \quad . \tag{9}$$

By Green's theorem, (7) reduces to

$$\theta = \int_V dv\ (\phi\ \nabla^2 G_i - G_i\ \nabla^2\phi) \quad , \tag{10}$$

which, by (4) and (9), reduces to

$$\theta = \int_V dv\ \left[\phi(-\frac{\omega^2}{v^2}\ G_i) - G_i(-\frac{\omega^2}{c^2(\mathbf{x},\omega)}\ \phi-\rho)\right] \quad , \tag{11}$$

$$\theta = \int_V dv\ G_i\ \rho + \int_V dv\ G_i \left[\frac{\omega^2}{c^2(\mathbf{x},\omega)} - \frac{\omega^2}{v^2}\right] \phi \quad . \tag{12}$$

(which, for the case of a known constant medium wave velocity c, unknown sources ρ, and a reference velocity chosen as $v=c$, reduces to the earlier inverse scattering integral equation of this author [1]).

SECTION A-III

SOLUTION OF THE INTEGRAL EQUATION

Let a medium reference wave slowness σ be introduced and defined as

$$\sigma \equiv \frac{1}{v} \quad . \tag{13}$$

The inverse scattering integral equation (12), in terms of this reference slowness, thus is

$$\theta(\mathbf{x},\omega,\sigma) = \int_V G_i(\mathbf{x}|\mathbf{x}',\omega,\sigma)\, \rho(\mathbf{x}',\omega)\, dv'$$

$$+ \int_V G_i(\mathbf{x}|\mathbf{x}',\omega,\sigma) \left[\frac{\omega^2}{c^2(\mathbf{x}',\omega)} - \omega^2 \sigma^2 \right] \psi(\mathbf{x}',\omega)\, dv' \quad , \tag{14}$$

where in two and three dimensions, the imaginary part of the free reference space Green's functions are

$$G_i(\mathbf{x}|\mathbf{x}',\omega,\sigma) = \tfrac{1}{4}\, J_o(\omega r\sigma) \tag{15}$$

and

$$G_i(\mathbf{x}|\mathbf{x}';\omega,\sigma) = \frac{e^{i\omega r\sigma}}{4\pi r} \tag{16}$$

respectively, and where $r \equiv |\mathbf{x}-\mathbf{x}'|$.

Taking the Hilbert transform of (14) with respect to the reference slowness yields

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\theta(\mathbf{x},\omega,\sigma')}{\sigma-\sigma'}\, d\sigma' = \int_V \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{G_i(\mathbf{x}|\mathbf{x}',\omega,\sigma')}{\sigma-\sigma'}\, d\sigma'\, \phi(\mathbf{x}',\omega)\, dv'$$

$$+ \int_V \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{G_i(\mathbf{x}|\mathbf{x}',\omega,\sigma')}{\sigma-\sigma'} \left[\frac{\omega^2}{c^2(\mathbf{x}',\omega)} - \omega^2 \sigma'^2 \right] d\sigma'\, \phi(\mathbf{x}',\omega)\, dv' \quad . \tag{17}$$

By (15) and (16), with the aid of [2,3] and twice repeated application of [4,5] *it follows that*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{G_i(\mathbf{x}|\mathbf{x}',\omega,\sigma')}{\sigma-\sigma'}\, d\sigma' = sgn(\omega r)\, G_r(\mathbf{x}|\mathbf{x}',\omega,\sigma) \quad , \tag{18}$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma'^2\, G_i(\mathbf{x}|\mathbf{x}',\omega,\sigma')}{\sigma-\sigma'}\, d\sigma' = sgn(\omega r)\, \sigma^2\, G_r(\mathbf{x}|\mathbf{x}',\omega,\sigma) \quad , \tag{19}$$

in two and three dimensions, but not in one dimension. Where in two and three dimensions G_r is the real part of the free reference space Green's function

$$G_r(\mathbf{x}|\mathbf{x}',\omega,\sigma) = -\frac{1}{4}\, Y_0(\omega r\sigma) \tag{20}$$

and

$$G_r(\mathbf{x}|\mathbf{x}',\omega,\sigma) = \frac{cos(\omega r\sigma)}{4\pi r} \tag{21}$$

respectively.

Thus, with the aid of (18) and (19), (17) becomes

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\theta(\mathbf{x},\omega,\sigma')}{\sigma-\sigma'}\, d\sigma' = \int_V sgn(\omega r)\; G_r(\mathbf{x}|\mathbf{x}',\omega,\sigma)\; \rho(\mathbf{x}',\omega)\; dv'$$

$$+ \int_V sgn(\omega r)\; G_r(\mathbf{x}|\mathbf{x}',\omega,\sigma) \left[\frac{\omega^2}{c^2(\mathbf{x}',\omega)} - \omega^2\,\sigma^2\right] \phi(\mathbf{x}',\omega)\; dv' \quad . \tag{22}$$

Restricting (22) to positive non-zero frequencies ω, permits its rewriting as the principal value integral (i.e., excluding integration over $\mathbf{x}=\mathbf{x}'$)

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\theta(\mathbf{x},\omega,\sigma')}{\sigma-\sigma'}\, d\sigma' = p\int_V G_r(\mathbf{x}|\mathbf{x}',\omega,\sigma)\; \rho(\mathbf{x}',\omega)\; dv'$$

$$+ \; p\int_V G_r(\mathbf{x}|\mathbf{x}',\omega,\sigma) \left[\frac{\omega^2}{c^2(\mathbf{x}',\omega)} - \omega^2\,\sigma^2\right] \phi(\mathbf{x}',\omega)\; dv' \quad . \tag{23}$$

Since the imaginary part of the free reference space Green's function is not singular at $\mathbf{x}=\mathbf{x}'$, it follows from the addition of (14) and (23) that for positive non-zero frequencies

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\theta(\mathbf{x},\omega,\sigma')}{\sigma-\sigma'}\, d\sigma' + i\theta(\mathbf{x},\omega,\sigma) = p\int_V G(\mathbf{x}|\mathbf{x}',\omega,\sigma)\, \rho(\mathbf{x}',\omega)\, dv'$$

$$+ p\int_V G(\mathbf{x}|\mathbf{x}',\omega,\sigma) \left[\frac{\omega^2}{c^2(\mathbf{x}',\omega)} - \omega^2\sigma^2 \right] \phi(\mathbf{x}',\omega)\, dv' \quad . \qquad (24)$$

For the inverse scattering inverse source case of a known constant medium velocity and unknown sources ρ (i.e., wave equation (1)), (24) reduces, after chosing the reference slowness $\sigma=\frac{1}{c}$, for $\omega>0$, to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\theta(\mathbf{x},\omega,\sigma)}{\frac{1}{c}-\sigma}\, d\sigma + i\theta(\mathbf{x},\omega,\frac{1}{c}) = p\int_V G(\mathbf{x}|\mathbf{x}',\omega,\frac{1}{c})\, \rho(\mathbf{x}',\omega)\, dv' \quad . \qquad (25)$$

At that reference slowness, the direct Kirchhoff integration (5) of the wave equation (1) can be written as

$$\phi(\mathbf{x},\omega) = p\int_V G(\mathbf{x}|\mathbf{x}',\omega,\frac{1}{c})\, \rho(\mathbf{x}',\omega)\, dv' + \phi_I(\mathbf{x},\omega) \quad , \qquad (26)$$

Since in two and three dimensions the Green's function singularity is weak and removable, and the Kirchhoff surface integral represents the incident field.

Thus combining (25) and (26) yields the solution

$$\phi(\mathbf{x},\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\theta(\mathbf{x},\omega,\sigma)}{\frac{1}{c}-\sigma}\, d\sigma + i\theta(\mathbf{x},\omega,\frac{1}{c}) + \phi_I(\mathbf{x},\omega) \quad , \quad \omega>0 \quad . \qquad (27)$$

For the inverse scattering inverse source case of known zero sources and unknown medium wave velocity $c(\mathbf{x},\omega)$, i.e., wave equation (3), (24) reduces, after choosing the reference slowness $\sigma=\frac{1}{c_o}$, for $\omega>0$, to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\theta(\mathbf{x},\omega,\sigma)}{\frac{1}{c_o}-\sigma}\, d\sigma + i\theta(\mathbf{x},\omega,\frac{1}{c_o})$$

$$= p\int_V G(\mathbf{x}|\mathbf{x}',\omega,\frac{1}{c_o}) \left[\frac{\omega^2}{c^2(\mathbf{x}',\omega)} - \frac{\omega^2}{c_o^2}\right] \phi(\mathbf{x}',\omega)\, dv' \quad . \tag{28}$$

A digression concerning the wave equation (3) is now in order. This wave equation can be rewritten as

$$\nabla^2\phi + \frac{\omega^2}{c_o^2}\phi = -\rho_o \quad , \tag{29}$$

where the sources ρ_o are reference sources relative to the arbitrarily chosen reference wave velocity c_o, given by the relative constitutive equation

$$\rho_o = V_o\phi \quad , \tag{30}$$

and the potential V_o is a reference potential relative to the arbitrarily chosen reference wave velocity c_o, given by

$$V_o = \frac{\omega^2}{c^2(\mathbf{x},\omega)} - \frac{\omega^2}{c_o^2} \quad . \tag{31}$$

It thus follows that (28) can be written for $\omega \neq 0$ as

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\theta(\mathbf{x},\omega,\sigma)}{\frac{1}{c_o}-\sigma}\, d\sigma + i\theta(\mathbf{x},\omega,\frac{1}{c_o}) = p\int_V G(\mathbf{x}|\mathbf{x}',\omega,\frac{1}{c_o})\, \rho_o(\mathbf{x}',\omega,\frac{1}{c_o})\, dv' \quad . \tag{32}$$

At this reference slowness $\frac{1}{c_o}$, the direct Kirchhoff integration (5) of the wave equation (29) can be written as

$$\phi(\mathbf{x},\omega) = p\int_V g(\mathbf{x}|\mathbf{x}',\omega,\frac{1}{c_o})\,\rho_s(\mathbf{x}',\omega,\frac{1}{c_o})\,dv' + \phi_i(\mathbf{x},\omega,\frac{1}{c_o}) \quad , \qquad (33)$$

since in two and three dimensions the Green's function singularity is weak and removable, and the Kirchhoff surface integral represents the incident field.

Thus, combining (32) and (33) yields the (same as (27)) solution

$$\phi(\mathbf{x},\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\phi(\mathbf{x},\omega,\sigma)}{\frac{1}{c_o}-\sigma}\,d\sigma + i\phi(\mathbf{x},\omega,\frac{1}{c_o}) + \phi_i(\mathbf{x},\omega,\frac{1}{c_o}) \quad , \qquad (34)$$

The sources, potential, and medium wave velocities can be obtained from knowledge of the fields in a variety of straight-forward manners.

One might be tempted to attempt to simplify the solutions (27,34) by applying and executing analytically the Hilbert transform with respect to the reference slowness dircectly on the surface integral definition (7) of the effectal field, thus obtaining this definition in terms of the principal value of the real part of the free reference space Green's function, instead of the imaginary part of this free reference space Green's function. The flaw with such an attempt is that on the surface of integration, away from the singular point of the real part of the free reference space Green's function, the principal value and the complete singular real part of the free reference space Green's function are indistinguishable and identical, and the application of Green's theorem as per (7-12) will, by the differentiability and continuity requirement imposed by Green's theorem, fail to generate the principal value of the real part of the free reference space Green's function in the volume interior to the surface of integration, which would have yielded the desired solution, but generate the full singular real part of the free reference space Green's function in this interior, which fails to yield the desired solution, and yield (a slightly modified) version of the integral equation (12), instead of its solution.

SECTION A-IV

APPENDIX REFERENCES

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END

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